

# String propagation near Kaluza-Klein black holes: an analytical and numerical study

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This paper presents a detailed investigation of the motion of a string near a Kaluza-Klein black hole, using the null string expansion. The zeroth-order string equations of motion are set up separately for electrically and magnetically charged black hole backgrounds. The case of a string falling head-on into the black hole is considered in detail. The equations reduce to quadratures for a magnetically charged hole, while they are amenable to numerical solution for an electrically charged black hole. The Kaluza-Klein radius seen by the string as it approaches the black hole decreases in the magnetic case and increases in the electric case. For magnetic backgrounds, analytical solutions can be obtained in terms of elliptical integrals. These reduce to elementary functions in special cases, including that of the well-known Pollard-Gross-Perry-Sorkin monopole. Here the string exhibits decelerated descent into the black hole. The results in the authors' earlier papers are substantiated here by presenting a detailed analysis. A preliminary analysis of first-order perturbations is also presented, and it is shown that the invariant string length receives a nonzero contribution in the first order.

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## I. INTRODUCTION

String theories [1,2] possess a much richer set of symmetries than theories of relativistic point particles. Classically, string theory is defined in any number of dimensions, but quantum considerations require it to be formulated in higher dimensions - 26 for bosonic strings and 10 for superstrings. To make a connection with physical four-dimensional physics, the extra dimensions are compactified. The mechanism of this compactification is an important issue in string theory, and needs to be studied using a variety of approaches.

The energy scales at which string theory is formulated are of the order of the Planck scale. However, it is not unreasonable to expect that there would be some residual effect at low energies, i.e. energies low compared to the Planck scale, but higher than any energies currently accessible at particle accelerators. One possible approach, in this spirit, could be to study string dynamics in curved spacetime [3] (see also [4]). The investigation is important in the context of understanding string quantisation in curved spacetime.

The string equations of motion are complicated and various ansatz and approximation schemes have been proposed to solve them. One such scheme is the null string expansion [5-7], which is suitable to describe string propagation in strong gravitational fields. A possible application of this investigation could be to study the dynamics of a string probe near a black hole where the extra dimensions are expected to contribute nontrivially.

Solving the higher dimensional string equations of motion is technically involved. An *in-between* approach is provided by studying a five-dimensional Kaluza-Klein background instead of the full  $D$ -dimensional spacetime (For a review of Kaluza-Klein theories, see [8]). It would be interesting to observe how the extra dimension ap-

pears from the point of view of a string falling into the black hole. An attempt in this direction was made by solving the string equations of motion in Kaluza-Klein black hole backgrounds [9,10]. It was shown that, even at the classical level, the extra compact dimension contributes nontrivially to string propagation. In this paper, we present a more detailed study of strings near Kaluza-Klein black holes. Our main tool is the null string expansion which is used to solve for the string coordinates [11]. The major results of the zeroth order investigations have already been reported in Refs. [9,10]. This paper provides the technical details involved in obtaining those results; in addition, we present for the first time some analysis of the first order equations.

In Section II, black hole solutions to five-dimensional Kaluza-Klein theory are reviewed. Section III reviews string propagation in curved spacetime, with string coordinates expanded in a perturbation series around the worldsheet velocity of light. Section IV is devoted to the study of string propagation in electrically and magnetically charged Kaluza-Klein black hole backgrounds. Section V presents analytical results for string propagation near extremal magnetically charged black holes. In Section VI, the analysis of Section IV and Section V is extended up to the first order in the perturbation expansion and some preliminary results presented. The concluding Section VII contains the summary and some remarks. Some technical details are presented in Appendix A.

## II. KALUZA-KLEIN BLACK HOLES

Stationary Kaluza-Klein solutions with spherical symmetry were studied systematically by Chodos and Detweiler [12] and Dobaish and Maison [13]. These black holes are characterised by the mass, the electric charge

and the scalar charge. It was shown by Gross and Perry [14] and in independent works by Pollard [15] and by Sorkin [16] that five-dimensional magnetic monopoles (electrically neutral) exist as solutions to five-dimensional Kaluza-Klein theories. The solutions in Ref. [12] were generalised by Gibbons and Wiltshire [17] to those with four parameters. It was shown that, in general, Kaluza-Klein black holes possess both electric and magnetic charge, with the solutions mentioned above as special cases (see also [18]).

We consider the metric background as given in [17]

$$ds^2 = -e^{4k\frac{\varphi}{\sqrt{3}}}(dx_5 + 2kA_\alpha dx^\alpha)^2 + e^{-2k\frac{\varphi}{\sqrt{3}}}g_{\alpha\beta}dx^\alpha dx^\beta, \quad (1)$$

where  $\varphi$  is the dilaton field,  $k^2 = 4\pi G$ ;  $x_5$  is the extra dimension and should be identified modulo  $2\pi R_0$ , where  $R_0$  is the radius of the circle about which the coordinate  $x_5$  winds.

The demand that the black hole solutions be regular in four dimensions (changing to units where  $G = 1$  [19]) implies

$$e^{4\varphi/\sqrt{3}} = \frac{B}{A}, \quad (2)$$

$$A_\alpha dx^\alpha = \frac{Q}{B}\left(r - \frac{\Sigma}{\sqrt{3}}\right)dt + P \cos \theta d\phi,$$

and

$$g_{\alpha\beta}dx^\alpha dx^\beta = \frac{f^2}{\sqrt{AB}}dt^2 - \frac{\sqrt{AB}}{f^2}dr^2 - \sqrt{AB}(d\theta^2 + \sin^2 \theta d\phi^2), \quad (3)$$

where  $A$ ,  $B$  and  $f^2$  depend on  $r$  and are given by

$$A = \left(r - \frac{\Sigma}{\sqrt{3}}\right)^2 - \frac{2P^2\Sigma}{\Sigma - \sqrt{3}M}, \quad (4)$$

$$B = \left(r + \frac{\Sigma}{\sqrt{3}}\right)^2 - \frac{2Q^2\Sigma}{\Sigma + \sqrt{3}M},$$

$$f^2 = (r - M)^2 - (M^2 + \Sigma^2 - P^2 - Q^2).$$

If  $P = Q = 0$ , we regain the usual Schwarzschild black holes.

The black hole solutions are characterised by the mass  $M$  of the black hole, the electric charge  $Q$ , the magnetic charge  $P$  and the scalar charge  $\Sigma$ . Out of the charges, only two are independent [17,19]. The constant parameters are constrained by the relation

$$\frac{2}{3}\Sigma = \frac{Q^2}{\Sigma + \sqrt{3}M} + \frac{P^2}{\Sigma - \sqrt{3}M}, \quad (5)$$

where the scalar charge is defined by

$$k\varphi \longrightarrow \frac{\Sigma}{r} + O\left(\frac{1}{r^2}\right) \text{ as } r \longrightarrow \infty.$$

Eq. 5 is invariant under the duality transformation

$$Q \longrightarrow P, \quad P \longrightarrow Q, \quad \Sigma \longrightarrow -\Sigma. \quad (6)$$

which relates "electric-like" and "magnetic-like" black holes. It is worth noting that physically distinct black holes are related by the duality.

The black hole solutions listed by Gibbons and Wiltshire include the much studied Pollard-Gross-Perry-Sorkin (PGPS) extremal magnetic black hole. Recently, it was shown that the PGPS monopole arises as a solution of a suitable dimensionally reduced string theory [20]. This provides an additional motivation to study such black hole backgrounds in the context of string theory. One way is by finding out stringy corrections to the five-dimensional black hole backgrounds [19]. A complementary approach is to study string propagation in Kaluza-Klein black hole backgrounds, which is the subject of this paper.

### III. NULL STRING EXPANSION

We start with the bosonic string worldsheet action [1] given by

$$S = -T_0 \int d\tau d\sigma \sqrt{-\det g_{ab}}, \quad (7)$$

where  $g_{ab} = G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu$  is the two-dimensional worldsheet metric;  $\sigma$  and  $\tau$  are the worldsheet coordinates.

The classical equations of motion are given by

$$\partial_\tau^2 X^\mu - c^2 \partial_\sigma^2 X^\mu + \Gamma_{\nu\rho}^\mu [\partial_\tau X^\nu \partial_\tau X^\rho - c^2 \partial_\sigma X^\nu \partial_\sigma X^\rho] = 0, \quad (8)$$

where  $\Gamma_{\nu\rho}^\mu$  are Christoffel symbols for the background metric.

The constraint equations are

$$\partial_\tau X^\mu \partial_\sigma X^\nu G_{\mu\nu} = 0 \quad (9)$$

$$[\partial_\tau X^\mu \partial_\tau X^\nu + c^2 \partial_\sigma X^\mu \partial_\sigma X^\nu] G_{\mu\nu} = 0. \quad (10)$$

String propagation in curved spacetime has been investigated in a number of papers [21–29]. Several simplifying ansatzes exist to solve the highly nonlinear string equations of motion. We follow the approach of de Vega and Nicolaidis [5], which uses the worldsheet velocity of light as an expansion parameter. The scheme involves systematic expansion in powers of  $c$ . The limit of small worldsheet velocity of light corresponds to that of small string tension. If  $c \ll 1$ , the coordinate expansion is suitable to describe strings in a strong gravitational background (see [6,7]). Here the derivatives w.r.t.  $\tau$  overwhelm the  $\sigma$  derivatives. In the opposite case ( $c \gg 1$ ),

the classical equations of motion give us a stationary picture as the  $\sigma$  derivatives dominate. The expansion for  $c = 1$  corresponds to the centre of mass expansion of the string.

We restrict ourselves to the case where  $c$  is small, our interest being to probe the dynamical behaviour of the extra dimensions. Using this expansion scheme, the string coordinates are expressed as

$$X^\mu(\sigma, \tau) = X_0^\mu(\sigma, \tau) + c^2 X_1^\mu(\sigma, \tau) + c^4 X_2^\mu(\sigma, \tau) + \dots \quad (11)$$

The zeroth order  $X_0^\mu(\sigma, \tau)$  satisfies the following set of equations:

$$\begin{aligned} \ddot{X}_0^\mu + \Gamma_{\nu\rho}^\mu \dot{X}_0^\nu \dot{X}_0^\rho &= 0, \\ \dot{X}_0^\mu \dot{X}_0^\nu G_{\mu\nu} &= 0, \\ \dot{X}_0^\mu X_0^{\nu\prime} G_{\mu\nu} &= 0. \end{aligned} \quad (12)$$

It is clear from the above equations that every point on the string moves along a null geodesic; thus the zeroth order equations describe the motion of a *null string* [5].

The first order fluctuations can be obtained by retaining terms of order  $c^2$  and the equations are given by [5]

$$\begin{aligned} \ddot{X}_1^\rho + 2\Gamma_{\kappa\lambda}^\rho \dot{X}_0^\lambda \dot{X}_1^\kappa + \Gamma_{\kappa\lambda, \alpha}^\rho \dot{X}_0^\kappa \dot{X}_0^\lambda X_1^\alpha \\ = X_0^{\prime\prime\rho} + \Gamma_{\kappa\lambda}^\rho X_0^{\prime\kappa} X_0^{\prime\lambda}, \end{aligned} \quad (13)$$

with the constraints being

$$(2\dot{X}_0^\mu \dot{X}_1^\nu + X_0^{\prime\mu} X_1^{\prime\nu}) G_{\mu\nu} + G_{\mu\nu, \alpha} \dot{X}_0^\mu \dot{X}_0^\nu X_1^\alpha = 0, \quad (14)$$

$$(\dot{X}_1^\mu X_0^{\prime\nu} + \dot{X}_0^\mu X_1^{\prime\nu}) G_{\mu\nu} + G_{\mu\nu, \alpha} \dot{X}_0^\mu X_0^{\prime\nu} X_1^\alpha = 0. \quad (15)$$

A physically interesting quantity is the invariant or proper string size  $l$ , which is given by [3]

$$dl^2 = X^{\prime\mu} X^{\prime\nu} G_{\mu\nu}(X) d\sigma^2. \quad (16)$$

The differential string size has the form of an effective mass for the geodesic motion [7]. At the zeroth order, the proper string length is indeterminate. At the first order and higher orders, string length varies as a string propagates in curved spacetime.

The motion of null strings in curved backgrounds has been studied in cosmological and black hole backgrounds [5–7]. Applying the formalism to FRW geometry, it was shown in [5] that the string expands or contracts at the same rate as the whole universe. Lousto and Sánchez [7] have made an extensive study of string propagation in conformally flat FRW spacetime and in black hole spacetimes. In the next Section we discuss propagation of a null string near Kaluza-Klein black holes.

## IV. STRING PROPAGATION IN KALUZA-KLEIN BLACK HOLE BACKGROUNDS

The Kaluza-Klein black hole, in general, has both magnetic and electric charges along with a scalar charge. Out of these three charges, two are independent as clearly shown in Eq. (5). We seek to solve the equations of motion for the string coordinates in the exterior of the black hole. For simplicity, we consider the magnetically and electrically charged cases separately. The main results obtained in this section have been reported earlier [9]; however, it contains technical details which have not been presented before.

### A. Magnetically charged black hole

The zeroth order equations of motion for the string coordinates are obtained by substituting the above metric in Eqs. (12). For an electrically neutral ( $Q = 0$ ) background the equations of motion are

$$\begin{aligned} \frac{\partial^2 t}{\partial \tau^2} + 2 \left( \frac{f'}{f} - \frac{B'}{2B} \right) \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} &= 0, \\ \frac{\partial^2 r}{\partial \tau^2} + \left[ -\frac{f^3}{2AB^2} (B'f - 2f'B) \right] \left( \frac{\partial t}{\partial \tau} \right)^2 \\ + \left( \frac{A'f - 2f'A}{2Af} \right) \left( \frac{\partial r}{\partial \tau} \right)^2 - \frac{f^2 A'}{2A} \left( \frac{\partial \phi}{\partial \tau} \right)^2 \\ + \frac{f^2}{2A^3} (A'B - B'A) \left( \frac{\partial x_5}{\partial \tau} \right)^2 &= 0, \\ \frac{\partial^2 \phi}{\partial \tau^2} + \frac{A'}{A} \left( \frac{\partial r}{\partial \tau} \right) \left( \frac{\partial \phi}{\partial \tau} \right) &= 0, \\ \frac{\partial^2 x_5}{\partial \tau^2} + \left( -\frac{A'}{A} + \frac{B'}{B} \right) \left( \frac{\partial r}{\partial \tau} \right) \left( \frac{\partial x_5}{\partial \tau} \right) &= 0. \end{aligned} \quad (17)$$

and the constraint equation is

$$\begin{aligned} \frac{f^2}{B} \left( \frac{\partial t}{\partial \tau} \right)^2 - \frac{A}{f^2} \left( \frac{\partial r}{\partial \tau} \right)^2 \\ - A \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{B}{A} \left( \frac{\partial x_5}{\partial \tau} \right)^2 &= 0. \end{aligned} \quad (18)$$

Here we have taken the string to be propagating in the equatorial plane, i.e.  $\theta = \pi/2$ . Hence the equation of motion for the coordinate  $\theta$  vanishes.

The functions  $A$ ,  $B$  and  $f$ , for the magnetically charged black hole case, are given by

$$\begin{aligned} A &= (r + \Sigma_1)(r - 3\Sigma_1), \\ B &= (r + \Sigma_1)^2, \\ f^2 &= (r + \Sigma_1)(r - 2M - \Sigma_1). \end{aligned} \quad (19)$$

where  $\Sigma_1 = \Sigma/\sqrt{3}$ .

The first integrals of motion are

$$\begin{aligned}
\frac{\partial t}{\partial \tau} &= \frac{c_1 B}{f^2}, \\
\frac{\partial \phi}{\partial \tau} &= \frac{c_2}{A}, \\
\frac{\partial x_5}{\partial \tau} &= c_3 \frac{A}{B}, \\
\left( \frac{\partial r}{\partial \tau} \right)^2 &= \frac{B}{A} c_1^2 - \frac{f^2}{A^2} c_2^2 - \frac{f^2}{B} c_3^2,
\end{aligned} \tag{20}$$

where  $c_1$ ,  $c_2$  and  $c_3$  are functions of  $\sigma$ . Since, at the zeroth order, only derivatives with respect to  $\tau$  are present, we can treat  $c_1$ ,  $c_2$  and  $c_3$  as constants. The constants  $c_1$  and  $c_2$  correspond respectively to the energy  $E(\sigma)$  and the angular momentum  $L(\sigma)$ . The first three equations are obtained by direct integration of the  $t$ ,  $\phi$  and  $x_5$  equations. The constraint equation (19) is then used to obtain the equation for  $\partial r / \partial \tau$ .

Since  $A$ ,  $B$  and  $f^2$  are all functions of  $r$ , it is convenient to change all the derivatives with respect to  $\tau$  to those with respect to  $r$ ,

$$\frac{\partial t}{\partial \tau} = \frac{dt}{dr} \frac{\partial r}{\partial \tau}, \quad \frac{\partial x_5}{\partial \tau} = \frac{dx_5}{dr} \frac{\partial r}{\partial \tau}, \quad \frac{\partial \phi}{\partial \tau} = \frac{d\phi}{dr} \frac{\partial r}{\partial \tau}. \tag{21}$$

Here we have assumed that the trajectory can be written in the parametric form,

$$t = t(r) \quad x_5 = x_5(r), \quad \phi = \phi(r) \tag{22}$$

From Eqs. (20) we have

$$\partial r / \partial \tau = \pm \sqrt{\frac{B}{A} c_1^2 - \frac{f^2}{A^2} c_2^2 - \frac{f^2}{B} c_3^2}$$

. We choose the negative sign as we consider an in-falling string.

This change of variables enables us to reduce the equations to quadratures:

$$\begin{aligned}
\tau &= - \int \frac{dr}{\sqrt{\frac{B}{A} c_1^2 - \frac{f^2}{B} c_3^2}}, \\
x_5 &= - \int \frac{c_3 A dr}{B \sqrt{\frac{B}{A} c_1^2 - \frac{f^2}{B} c_3^2}}, \\
t &= - \int \frac{c_1 B dr}{f^2 \sqrt{\frac{B}{A} c_1^2 - \frac{f^2}{B} c_3^2}},
\end{aligned} \tag{23}$$

up to constants of integration which depend on  $\sigma$ . Here we have taken  $c_2 = 0$ , i.e. the string is falling in ‘head-on’. The quadratures can be solved numerically to obtain  $t$ ,  $r$ , and  $x_5$  as functions of  $\tau$ . It is clear from Eq. (5) that, for  $P^2$  to be positive, we have

$$\Sigma < 0 \quad \text{or} \quad \Sigma > \sqrt{3}M. \tag{24}$$

The integrals (23) are evaluated numerically and inverted to obtain the coordinates as functions of  $\tau$ . We

confine ourselves to the region  $r > M$  and we assume  $r \gg \Sigma$ , i.e., from the integrals we drop terms of  $O(\Sigma^2/r^2)$ . The behaviour of  $r$  as a function of  $\tau$  is shown in Fig. 1 for different values of  $c_1$  and  $c_3$ .

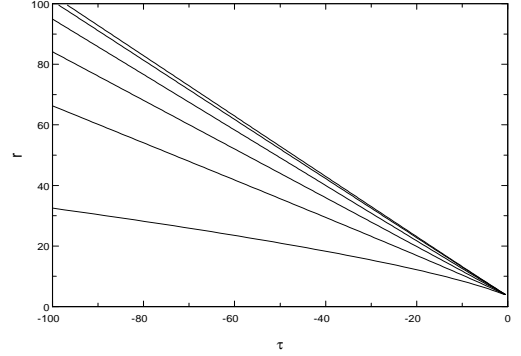


FIG. 1.  $r$  versus  $\tau$  for magnetically charged black hole.

## B. Electrically charged black hole

For the electrically charged ( $P = 0$ ) black hole, the equations of motion in the zeroth order take the form

$$\begin{aligned}
&AB[Af^2 + 12Q^2(r - \Sigma_1)^2] \frac{\partial^2 t}{\partial \tau^2} \\
&\quad - [12A'BQ^2(r - \Sigma_1)^2 + B'A^2f^2 \\
&\quad - 4B'AQ^2(r - \Sigma_1)^2 - 2f'A^2Bf \\
&\quad - 8ABQ^2(r - \Sigma_1)] \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} \\
&\quad + 4QAB[B'(r - \Sigma_1) - B] \frac{\partial r}{\partial \tau} \frac{\partial x_5}{\partial \tau} = 0, \\
&2A^3B^2f \frac{\partial^2 r}{\partial \tau^2} + f^3[4A'BQ^2(r - \Sigma_1)^2 - B'A^2f^2 \\
&\quad + 4B'AQ^2(r - \Sigma_1)^2 + 2f'A^2Bf \\
&\quad - 8ABQ^2(r - \Sigma_1)] \left( \frac{\partial t}{\partial \tau} \right)^2 \\
&\quad - 8f^3B^2Q[A'(r - \Sigma_1) - A] \frac{\partial t}{\partial \tau} \frac{\partial x_5}{\partial \tau} \\
&\quad + A^2B^2[A'f - 2f'A] \left( \frac{\partial r}{\partial \tau} \right)^2
\end{aligned} \tag{25}$$

$$\begin{aligned}
& -A^2 B^2 f^3 A' \left( \frac{\partial \phi}{\partial \tau} \right)^2 \\
& + f^3 B^2 [A'B - B'A] \left( \frac{\partial x_5}{\partial \tau} \right)^2 = 0, \\
& AB^2 [Af^2 + 12Q^2(r - \Sigma_1)^2] \frac{\partial^2 x_5}{\partial \tau^2} \\
& - 4QA[A'Bf^2(r - \Sigma_1) - B'Af^2(r - \Sigma_1) \\
& + 4B'Q^2(r - \Sigma_1)^3 + 2f'ABf(r - \Sigma_1) \\
& - ABf^2 - 4BQ^2(r - \Sigma_1)^2] \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} \\
& - B[A'ABf^2 + 12A'BQ^2(r - \Sigma_1)^2 \\
& - B'A^2f^2 + 4B'AQ^2(r - \Sigma_1)^2 \\
& - 16ABQ^2(r - \Sigma_1)] \frac{\partial r}{\partial \tau} \frac{\partial x_5}{\partial \tau} = 0.
\end{aligned}$$

and the constraint equation is

$$\begin{aligned}
& \left\{ \frac{f^2}{B} - \frac{4Q^2}{AB}(r - \Sigma_1)^2 \right\} \left( \frac{\partial t}{\partial \tau} \right)^2 \\
& - \frac{8Q}{A}(r - \Sigma_1) \frac{\partial t}{\partial \tau} \frac{\partial x_5}{\partial \tau} - \frac{A}{f^2} \left( \frac{\partial r}{\partial \tau} \right)^2 \\
& - A \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{B}{A} \left( \frac{\partial x_5}{\partial \tau} \right)^2 = 0,
\end{aligned} \tag{26}$$

where  $\Sigma_1 = \frac{\Sigma}{\sqrt{3}}$ .

In this case, the functions  $A$ ,  $B$  and  $f$  (which appear in the metric) are

$$\begin{aligned}
A &= (r - \Sigma_1)^2, \\
B &= (r - \Sigma_1)(r + 3\Sigma_1), \\
f^2 &= (r - \Sigma_1)(r - 2M - \Sigma_1).
\end{aligned} \tag{27}$$

The structure of the equations of motion, in this case, is such that they are not reducible to quadratures and we have to solve the differential equations numerically. Again, we consider an in-falling string in the region where  $r \gg \Sigma$  and  $\theta = \pi/2$ .

In this limit, we drop terms of  $O(\Sigma_1^2/r^2)$  and obtain the following equations

$$\begin{aligned}
& (r^2 - 2Mr + 26\Sigma_1 M) \frac{\partial^2 t}{\partial \tau^2} \\
& + \frac{2(Mr + \Sigma_1 r - 12\Sigma_1 M)}{r} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} + 4Q \frac{\partial r}{\partial \tau} \frac{\partial x_5}{\partial \tau} = 0, \\
& (r^2 - 2Mr + 26\Sigma_1 M) \frac{\partial^2 x_5}{\partial \tau^2} \\
& - \frac{4Q(r + 2\Sigma_1)}{r} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} \\
& - \frac{4\Sigma_1(r^2 + 6Mr + 2\Sigma_1 M)}{r^2} \frac{\partial r}{\partial \tau} \frac{\partial x_5}{\partial \tau} = 0,
\end{aligned} \tag{28}$$

with the constraint equation reducing to

$$\begin{aligned}
\left( \frac{\partial r}{\partial \tau} \right)^2 &= \frac{1}{r^3} (r^3 - 4Mr^2 - 4M\Sigma_1 r + 4M^2 r \\
& + 4M^2 \Sigma_1 + 12M^2 \Sigma_1) \left( \frac{\partial t}{\partial \tau} \right)^2 \\
& - \frac{8Q}{r^3} (r^2 - 2Mr + 3\Sigma_1 r - 4M\Sigma_1) \frac{\partial t}{\partial \tau} \frac{\partial x_5}{\partial \tau} \\
& - \frac{1}{r^2} (r^2 + 6\Sigma_1 r - 2Mr - 10\Sigma_1 M) \left( \frac{\partial x_5}{\partial \tau} \right)^2.
\end{aligned} \tag{29}$$

The leading-order analysis of Eqs.(28) and Eq.(29), which is required for numerical solution, is rather involved. Only the main results are presented here. Details can be found in Appendix A. To the leading order in powers of  $r$ , the equations can be written as

$$\begin{aligned}
& r^2 \frac{d\psi_1}{dr} - (\Sigma_1 + M)\psi_1^3 + 3(\Sigma_1 + M)\psi_1 \\
& \mp 4Q\sqrt{\psi_1^2 - 1}(\psi_1 - 1) = 0, \\
& r^2 \frac{d\psi_2}{dr} - 6\Sigma_1\psi_2^3 + (3\Sigma_1 + M)\psi_2 \\
& \mp 4Q\sqrt{\psi_2^2 + 1}(\psi_2 + 1) = 0,
\end{aligned} \tag{30}$$

where  $\psi_1 = \frac{dt}{dr}$ ,  $\psi_2 = \frac{dx_5}{dr}$  and  $\psi_1^2 - \psi_2^2 = 1$ . Only terms with highest power of  $r$  are retained to find out the large  $r$  behaviour of the solutions.

These decoupled equations can be solved to find

$$\begin{aligned}
\psi_1 &\equiv \frac{dt}{dr} = \frac{-1}{\sqrt{\mathcal{A}\sqrt{2M/r + c_1}}}, \\
\psi_2 &\equiv \frac{dx_5}{dr} = \frac{-1}{\sqrt{2M/r + c_2}},
\end{aligned} \tag{31}$$

where  $\mathcal{A}$  and  $c_1$  are constants with  $c_1 = \frac{1-A}{A}$ , and consequently, substituting in Eq. (29), we have (see Appendix A)

$$\frac{\partial r}{\partial \tau} = I \frac{e^{-M/r}}{(2M/r + c_1)^{1/2\mathcal{A}}}. \tag{32}$$

Therefore, the derivatives of  $t$  and  $x_5$  w.r.t  $\tau$  (in leading order) can be written as

$$\frac{\partial t}{\partial \tau} = \frac{-I}{\sqrt{\mathcal{A}\sqrt{2M/r + c_1}}} \frac{e^{-M/r}}{(2M/r + c_1)^{1/2\mathcal{A}}} \equiv f(r), \tag{33}$$

$$\frac{\partial x_5}{\partial \tau} = \frac{-I}{\sqrt{\mathcal{A}\sqrt{2M/r + c_2}}} \frac{e^{-M/r}}{(2M/r + c_1)^{1/2\mathcal{A}}} \equiv g(r). \tag{34}$$

The negative sign is because  $r$  decreases with  $\tau$  for an in-falling string.

The equations (28) are transformed to a convenient form by changing variables in the following manner

$$\begin{aligned}
\frac{\partial t}{\partial \tau} &= f_1(r)f(r), \\
\frac{\partial x_5}{\partial \tau} &= f_2(r)g(r).
\end{aligned} \tag{35}$$

Substituting in Eqs. (28) and defining  $u = 1/r$ , we have

$$\begin{aligned}\frac{df_1}{du} &= \frac{-1}{(1 - 2Mu + 26\Sigma_1 Mu^2)} \left[ -4Qf_2 \frac{g}{f} \right. \\ &\quad \left. - 2(M + \Sigma_1 - 12\Sigma_1 Mu)f_1 \right. \\ &\quad \left. + (1 - 2Mu + 26M\Sigma_1 u^2) \frac{df}{du} \frac{f_1}{f} \right] \\ \frac{df_2}{du} &= \frac{-1}{(1 - 2Mu + 26\Sigma_1 Mu^2)} \left[ 4Q(1 + 2\Sigma_1 u)f_1 \frac{f}{g} \right. \\ &\quad \left. + 4\Sigma_1(1 + 6Mu + 2\Sigma_1 Mu^2)f_2 \right. \\ &\quad \left. + (1 - 2Mu + 26M\Sigma_1 u^2) \frac{dg}{du} \frac{f_2}{g} \right].\end{aligned}\quad (36)$$

The set of Eqs. (36) and Eq. (29) have been solved numerically to obtain the coordinates as functions of  $\tau$ , the initial conditions for  $f_1$  and  $f_2$  being

$$\begin{aligned}f_1(u \rightarrow 0) &\longrightarrow 1, \\ f_2(u \rightarrow 0) &\longrightarrow 1.\end{aligned}\quad (37)$$

Again we have a two-parameter family of solutions. Fig. 2 illustrates how  $r$  varies as functions of  $\tau$  for different choices of integration constants. Here, the constant  $c_1$  is kept fixed at 1 and  $c_2$  is varied.

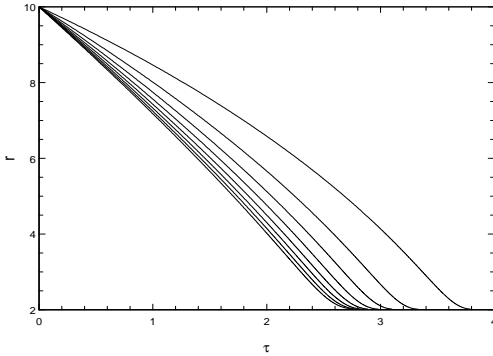


FIG. 2.  $r$  versus  $\tau$  for electrically charged black hole.

### C. Kaluza-Klein radius

As mentioned earlier, the interest lies in seeing the effect of the extra dimension on string propagation. The behaviour of the extra dimension is different in the two cases [9]. However, the picture is easier to interpret if we study the Kaluza-Klein radius, which is related to its asymptotic value  $R_0$  as

$$R(r) = R_0 \left( \frac{B}{A} \right)^{1/2}. \quad (38)$$

The radius  $R(r)$  has an implicit dependence on  $\tau$  through  $R(\tau) = R(r(\tau))$ , and is hence a dynamical quantity.

The effect of the magnetic field is to shrink the extra dimension (as already indicated in [17]), i.e., as the string approaches the black hole, the value of the Kaluza-Klein radius which it sees becomes smaller than its asymptotic value. The presence of electric charge tends to expand the extra dimension. The opposite behaviour in the two cases was illustrated in Ref. [9] and it was shown that even at the classical level, there is a nontrivial contribution of the extra dimension on string propagation.

## V. ANALYTICAL RESULTS FOR STRINGS NEAR MAGNETIC BLACK HOLES

### A. General Analytical Solutions for Magnetically Charged Black Hole

The quadratures (23) can be solved to obtain analytical counterparts of the solutions presented in the previous Section for the magnetically charged black hole. The integrals can be reduced to combinations of elliptical integrals, depending on the relative values of the constants  $\Sigma_1$  and  $M$  (see [30]). As an illustrative example we consider the case when  $c_1^2 = c_3^2$ . We can then write the integrals as

$$\begin{aligned}\tau &= \frac{1}{c_1} \int dr \frac{\sqrt{(r - 3\Sigma_1)(r + \Sigma_1)}}{\sqrt{(r - \alpha)}} \\ x_5 &= \int dr \frac{(r - 3\Sigma_1)^{3/2}}{\sqrt{(r + \Sigma_1)(r - \alpha)}} \\ t &= \int dr \frac{(r + \Sigma_1)^{3/2} \sqrt{(r - 3\Sigma_1)}}{(r - 2M - \Sigma_1) \sqrt{(r - \alpha)}}\end{aligned}\quad (39)$$

where  $\alpha = \frac{\Sigma_1(\Sigma_1 + 3M)}{(3\Sigma_1 + M)}$ .

As mentioned above, for  $Q^2$  to be positive, the scalar charge  $\Sigma_1$  should either be negative or greater than  $M$ . We first study the magnetic background with a negative scalar charge. We write  $\Sigma_1$  as

$$\Sigma_1 = -e \quad (40)$$

where  $e$  is positive. If  $\frac{e(e-3M)}{(M-3e)} < e$ , the solutions (modulo integration constants which depend on  $\sigma$ ) are given by

$$\begin{aligned}
\tau &= \frac{2}{3} \sqrt{(r+3e)(r-e)} \left( r - \frac{e(e-3M)}{(3e-M)} \right) \\
&+ \frac{16}{3} e \sqrt{e} \frac{(M+e)}{(M-3e)} E \left( f(r), \frac{-2e}{M-3e} \right) \\
&+ \frac{16}{3} e \sqrt{e} \frac{(M-e)}{(M-3e)} F \left( f(r), \frac{-2e}{M-3e} \right) \\
x_5 &= \frac{2}{3} \sqrt{(r+3e)(r-e)} \left( r - \frac{e(e-3M)}{(3e-M)} \right) \\
&- \frac{8e + \sqrt{e}(5M-11e)}{3(M-3e)} E \left( f(r), \frac{-2e}{M-3e} \right) \\
&+ \frac{4e\sqrt{e}(11M-25e)}{(M-3e)} F \left( f(r), \frac{-2e}{M-3e} \right) \\
t &= \frac{2}{3} \sqrt{(r+3e)(r-e)} \left( r - \frac{e(e-3M)}{(3e-M)} \right) \\
&+ \frac{8\sqrt{e}}{3} \left[ \frac{2e^2 + 11Me - 3M^2}{M-3e} \right] E \left( f(r), \frac{-2e}{M-3e} \right) \\
&- \frac{4M}{\sqrt{e}} \Pi \left( -\frac{4e^2}{(M+2e)(M-3e)}, f(r), \frac{-2e}{M-3e} \right)
\end{aligned} \tag{41}$$

where  $f(r) = \text{Arcsin} \left[ \sqrt{\frac{(r+3e)(3e-M)}{2e^2}} \right]$ . In these expressions  $F$ ,  $E$  and  $\Pi$  are elliptical functions of the first, second and third kind respectively [30,31].

If  $\frac{e(e-3M)}{(M-3e)} > e$ , the solutions are

$$\begin{aligned}
\tau &= \frac{2}{3} \sqrt{(r+3e)(r-e)} \left( r - \frac{e(e-3M)}{(3e-M)} \right) \\
&+ \frac{16}{3} \frac{\sqrt{2}e^2}{\sqrt{3e-M}} \frac{(M+e)}{(M-3e)} E \left( g(r), \frac{-(M-3e)}{2e} \right) \\
&+ \frac{2}{3} \frac{\sqrt{2}e^2}{\sqrt{3e-M}} \frac{(M+e)}{(M-3e)} F \left( g(r), \frac{-(M-3e)}{2e} \right) \\
x_5 &= \frac{2}{3} \sqrt{(r+3e)(r-e)} \left( r - \frac{e(e-3M)}{(3e-M)} \right) \\
&- \frac{8e\sqrt{2}e(5M-11e)}{3(M-3e)} \frac{\sqrt{3M-5e}}{\sqrt{M-3e}} E \left( g(r), \frac{-(M-3e)}{2e} \right) \\
&+ \frac{2e\sqrt{2}e(13M-23e)}{3(M-3e)} F \left( g(r), \frac{-(M-3e)}{2e} \right) \\
t &= \frac{2}{3} \sqrt{(r+3e)(r-e)} \left( r - \frac{e(e-3M)}{(3e-M)} \right) \\
&- \frac{8\sqrt{2}e}{3\sqrt{3e-M}} (3M^2 - 2e^2 - 11Me) \\
&\times E \left( g(r), \frac{-(M-3e)}{2e} \right)
\end{aligned} \tag{42}$$

$$\begin{aligned}
&- \frac{4e\sqrt{2}}{3\sqrt{3e-M}} \left[ \frac{-2e(e-3M)}{(M-3e)^2} (e^2 - 16Me + 3M^2) \right. \\
&+ 12e^2 - 8Me - 12M^2 \left. \right] F \left( g(r), \frac{-(M-3e)}{2e} \right) \\
&- \frac{4M^2\sqrt{3e-M}}{\sqrt{2e}} \Pi \left[ \frac{2e}{M-2e}, g(r), \frac{-(M-3e)}{2e} \right],
\end{aligned}$$

where,  $g(r) = \sqrt{\frac{r+3e}{4e}}$ . The case  $\Sigma_1 > M$  can be treated in a very similar manner; we omit the general expressions. A specific example, for the extremal black hole with  $\Sigma_1 = 2M$  is discussed in the next subsection.

The solutions reduce to elementary functions in the region where  $r$  is very large compared to the scalar charge. We take for instance the case when  $c_1 = c_3 = 1$ . Up to the first order in  $\Sigma_1/r$ , the solutions are

$$\begin{aligned}
\tau &= -\frac{2}{3\sqrt{2(M+\Sigma_1)}} \left\{ r + \frac{M\Sigma_1}{M+3\Sigma_1} \right\}^{3/2} \\
x_5 &= -\frac{2}{\sqrt{2(M+\Sigma_1)}} \left[ \left( \frac{r}{3} + \alpha - \frac{9\Sigma_1}{2} \right) (r - \Sigma_1)^{1/2} \right. \\
&- \frac{(7\Sigma_1 - \alpha)^{3/2}}{\sqrt{2(M+\Sigma_1)}} \tan^{-1} \left\{ \frac{\sqrt{2(r-3\Sigma_1)}}{\sqrt{7\Sigma_1 - \alpha}} \right\} \\
t &= -\frac{2}{\sqrt{2(M+\Sigma_1)}} \left[ (r - \alpha) \left\{ \frac{r}{3} + 2M + \Sigma_1 \right\} \right. \\
&+ \frac{2(2M + \Sigma_1)^2}{\sqrt{2(M+\Sigma_1)}\sqrt{\alpha - 2M - \Sigma_1}} \\
&\times \tan^{-1} \left\{ \frac{\sqrt{r - \alpha}}{\sqrt{\alpha - 2M - \Sigma_1}} \right\}
\end{aligned} \tag{43}$$

where  $\Sigma_1 = \Sigma/\sqrt{3}$  and  $\alpha = \frac{3M\Sigma_1}{3\Sigma_1 + M}$ . These solutions are valid in the region outside the horizon but not asymptotically far from the black hole. The negative sign comes because we consider an in-falling string. These solutions match with the numerical solutions presented in the last section, for the corresponding values of  $c_1$  and  $c_3$  [10].

## B. String Equations in Extremal Black Hole backgrounds

In the last sub-section, we discussed analytical solutions of the equations of motion of a string propagating in a magnetically charged black hole background. The solutions reduce to elementary functions in a suitable large distance approximation, i.e., the scalar charge is very small compared to the distance  $r$ . In fact, the black hole backgrounds in this case can be thought of as being small deviations from the Schwarzschild black holes. The numerical results of the previous Section were also obtained in this limit.

However, the integrals of motion (23) can be solved analytically without resorting to the limit  $r \gg \Sigma$ , if  $P = 2M$  and  $Q = 0$ , i.e, for an extremal magnetically

charged black hole. The constraint Eq. (5) implies that  $\Sigma_1 = -M$  or  $\Sigma_1 = 2M$ . The former case is that of the much-studied Pollard-Gross-Perry-Sorkin monopole [15,14,16]. In that case, the metric reduces to the form reported in Ref. [14]. The solutions are

$$\begin{aligned}\tau = t &= \frac{1}{\sqrt{c_1^2 - c_3^2}} (r - \beta M)^{1/2} (r + 3M)^{1/2} \\ &+ \frac{(3 + \beta)M}{\sqrt{c_1^2 - c_3^2}} \ln \left[ (r - \beta M)^{1/2} + (r + 3M)^{1/2} \right] \\ x_5 &= \frac{1}{\sqrt{c_1^2 - c_3^2}} (r - \beta M)^{1/2} (r + 3M)^{1/2} \\ &+ \frac{(11 + \beta)M}{\sqrt{c_1^2 - c_3^2}} \ln \left[ (r - \beta M)^{1/2} + (r + 3M)^{1/2} \right] \\ &+ \frac{16M}{\sqrt{\beta - 1} \sqrt{c_1^2 - c_3^2}} \left[ \arctan \frac{2\sqrt{r - \beta M}}{\sqrt{\beta - 1} \sqrt{r + 3M}} \right]\end{aligned}\quad (44)$$

where  $\beta = \frac{c_1^2 + 3c_3^2}{c_1^2 - c_3^2}$ . We choose  $c_1 = 1$ ; the condition of reality of the solutions then forces  $c_3 < 1$  and consequently  $\beta > 1$ . Here the time coordinate  $t$  is the same as the proper time  $\tau$  of the string, as in this case  $f^2/B = 1$ .

Fig. 3 shows a plot of  $r$  versus  $\tau$  for  $c_3 = 0.6$ . A comparison of Figs. 1 and 3 shows that the approach to the horizon is different in the two cases. In the PGPS case, the string decelerates as it approaches the horizon. This is not surprising, as the ‘repulsive’ or ‘anti-gravity’ effect of extremal black holes has been commented on in the literature (see, for example, [32] and [33]). The effect of the gauge field is opposite to that of gravity.

In addition to the above case, there is another extremal black hole solution (which has not been mentioned hitherto in the literature) corresponding to  $\Sigma_1 = 2M$ . The integrals can be solved in terms of elliptical functions, the solutions being

$$\begin{aligned}\tau &= \frac{1}{3} \sqrt{\frac{2}{7M}} \sqrt{(r - 6M)(r + 2M)} \left( r - \frac{10M}{7} \right) \\ &- \frac{32M}{21\sqrt{7}} \left[ \left\{ E\left(g(r), \frac{3}{7}\right) - 4F\left(g(r), \frac{3}{7}\right) \right\} \right] \\ x_5 &= \frac{1}{3} \sqrt{\frac{2}{7M}} \sqrt{(r - 6M)(r + 2M)} \left( r - \frac{10M}{7} \right) \\ &- \frac{32M}{21\sqrt{7}} \left[ \left\{ 22 E\left(g(r), \frac{3}{7}\right) - 4F\left(g(r), \frac{3}{7}\right) \right\} \right] \\ t &= \frac{1}{3} \sqrt{\frac{2}{7M}} \sqrt{(r - 6M)(r + 2M)} \left( r - \frac{10M}{7} \right) \\ &- \frac{2M}{21\sqrt{7}} \left[ 52 F\left(g(r), \frac{3}{7}\right) \right] \\ &+ \frac{M}{21\sqrt{7}} \left[ 59 \left\{ 8M E\left(g(r), \frac{3}{7}\right) - 6M F\left(g(r), \frac{3}{7}\right) \right\} \right] \\ &+ \frac{2M}{21\sqrt{7}} \left[ 63 \Pi\left(\frac{4}{7}, g(r), \frac{3}{7}\right) \right],\end{aligned}\quad (45)$$

where  $g(r) = \arcsin \left[ \frac{1}{2} \sqrt{\frac{7}{6}} \sqrt{\frac{r + 2M}{M}} \right]$ . We choose  $c_1 = c_3 = 1$ .

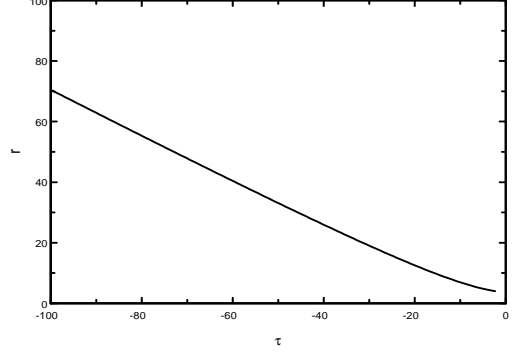


FIG. 3.  $r$  versus  $\tau$  when  $\Sigma_1 = -M$  for  $c_1 = 1$  and  $c_3 = 0.6$ .

## VI. FIRST ORDER PERTURBATIONS

The effect of the background geometry on the string probe itself is manifest at the first order and at higher orders. For the magnetically charged black hole background, the string equations of motion, to first order, are given by

$$\begin{aligned}\ddot{B}^0 + \Gamma_{01}^0 (\dot{A}^0 \dot{B}^1 + \dot{A}^1 \dot{B}^0) + 2\Gamma_{01,1}^0 \dot{A}^0 \dot{A}^1 B^1 \\ = A''^0 + 2\Gamma_{01}^0 A'^0 A'^1 \\ \ddot{B}^1 + 2\Gamma_{11}^1 \dot{A}^1 \dot{B}^1 B^1 + 2\Gamma_{55}^1 \dot{A}^5 \dot{B}^5 + \Gamma_{11,1}^1 \dot{A}^{1^2} B^1 \\ + \Gamma_{55,1}^1 \dot{A}^{5^2} B^1 = A''^1 + \Gamma_{11}^1 A'^1 A'^2 + \Gamma_{55}^1 A'^5 A'^2 \\ \ddot{B}^5 + 2\Gamma_{51}^5 (\dot{A}^5 \dot{B}^1 + \dot{A}^1 \dot{B}^5) + 2\Gamma_{51,1}^5 \dot{A}^5 \dot{A}^1 B^1 \\ = A''^5 + 2\Gamma_{51}^5 A'^5 A'^1\end{aligned}\quad (46)$$

and the constraints are

$$\begin{aligned}(2\dot{A}^0 \dot{B}^0 + A''^{0^2})G_{00} + (2\dot{A}^1 \dot{B}^1 + A''^{1^2})G_{11} \\ + (2\dot{A}^4 \dot{B}^5 + A''^{5^2})G_{55} \\ + G_{00,1} \dot{A}^{0^2} B^1 + G_{11,1} \dot{A}^{1^2} B^1 + G_{55,1} \dot{A}^{5^2} B^1 = 0. \\ (\dot{B}^0 A'^0 + \dot{A}^0 B'^0)G_{00} + (\dot{B}^1 A'^1 + \dot{A}^1 B'^1)G_{11} \\ + (\dot{B}^5 A'^5 + \dot{A}^5 B'^5)G_{55} + G_{00,1} \dot{A}^0 A'^0 B^1 \\ + G_{55,1} \dot{A}^1 A'^1 B^1 + G_{55,1} \dot{A}^5 A'^5 B^1 = 0.\end{aligned}\quad (47)$$



Here, for convenience, we have defined  $A \equiv X_0$  and  $B \equiv X_1$ . Since we confine the string to propagate in the equatorial plane and to fall in head-on, the  $B^2$  and  $B^3$  equations vanish.

At the first order in  $c^2$ , the equations of motion are second-order coupled partial differential equations. The right-hand sides of Eq. (46) involve  $\sigma$ -derivatives, while the left-hand sides, which contain the unknown functions  $B^\mu$ , involve  $\tau$  derivatives. Thus in principle the equations can be solved like ordinary differential equations for a fixed  $\sigma$ . In this Section, we consider an illustrative example without attempting to numerically solve the equations of motion.

We take the simplest case, where  $\Sigma_1 = -M$ , i.e., the PGPS monopole. The first order equations, in this case, are

$$\ddot{B}^0 = A''^0, \quad (48)$$

$$\begin{aligned} \ddot{B}^1 + \frac{4M}{(r-M)(r+3M)} \left\{ \sqrt{\frac{(r-M)}{(r+3M)}} c_1^2 - c_3^2 \right\} \dot{B}^1 & \quad (49) \\ + \frac{4M(r+M)}{(r-M)(r+3M)^3} c_1^2 B^1 \\ - \frac{16M^2}{(r-M)^2(r+3M)^2} c_3^2 B^1 - \frac{4M}{(r+3M)^2} c_3 \dot{B}^5 \\ = A''^1 - \frac{2M}{(r-M)(r+3M)} A'^1{}^2 - \frac{2M(r-M)}{(r+3M)^3} A'^5{}^2, \end{aligned}$$

$$\begin{aligned} \ddot{B}^5 - \frac{4M}{(r-M)(r+3M)} \left\{ \sqrt{\frac{(r-M)}{(r+3M)}} c_1^2 - c_3^2 \right\} \dot{B}^5 & \quad (50) \\ + \frac{4M}{(r-M)^2} c_3 \dot{B}^1 \\ + \frac{8M(r+M)}{(r-M)^3} c_3 \left\{ \sqrt{\frac{(r-M)}{(r+3M)}} c_1^2 - c_3^2 \right\} B^1 \\ = A''^5 + \frac{4M}{(r-M)(r+3M)} A'^1 A'^5, \end{aligned}$$

with the constraint Eq. (14) reducing to

$$\begin{aligned} 2c_1 \dot{B}^0 + \frac{2(r+3M)}{(r-M)} \left\{ \sqrt{\frac{(r-M)}{(r+3M)}} c_1^2 - c_3^2 \right\} \dot{B}^1 & \quad (51) \\ - 2c_3 \dot{B}^5 + \frac{4M}{(r-M)(r+3M)} c_1^2 B^1 \\ - \frac{8M}{(r-M)^2} c_3^2 B^1 + A'^0{}^2 - \frac{(r+3M)}{(r-M)} A'^1{}^2 \\ - \frac{(r-M)}{(r+3M)} A'^5{}^2 = 0. \end{aligned}$$

where  $r = A^1$ .

Rewriting Eqs. (44), including the integration constants explicitly, we have

$$\begin{aligned} \tau + I_1 &= \frac{1}{c_1} (A^0 + I_0) \quad (52) \\ &= \frac{1}{\sqrt{c_1^2 - c_3^2}} (A^1 - \beta M)^{1/2} (A^1 + 3M)^{1/2} \\ &+ \frac{(3+\beta)M}{\sqrt{c_1^2 - c_3^2}} \ln \left[ (A^1 - \beta M)^{1/2} + (A^1 + 3M)^{1/2} \right] \\ x_5 + I_5 &= \frac{1}{\sqrt{c_1^2 - c_3^2}} (A^1 - \beta M)^{1/2} (A^1 + 3M)^{1/2} \\ &+ \frac{(11+\beta)M}{\sqrt{c_1^2 - c_3^2}} \ln \left[ (A^1 - \beta M)^{1/2} + (A^1 + 3M)^{1/2} \right] \\ &+ \frac{16M}{\sqrt{\beta-1}\sqrt{c_1^2 - c_3^2}} \left[ \arctan \frac{2\sqrt{A^1 - \beta M}}{\sqrt{\beta-1}\sqrt{A^1 + 3M}} \right]. \end{aligned}$$

The constant  $c_1$  corresponds to the energy  $E(\sigma)$ . We choose  $c_1$  and  $c_3$  to be constants independent of  $\sigma$ . The form of the equations depends on how the functions  $I_0$ ,  $I_1$  and  $I_5$  depend on  $\sigma$ . We compute the derivatives of  $A^0$ ,  $A^1$  and  $A^5$ , viz.

$$\begin{aligned} \frac{\partial A^0}{\partial \sigma} &= -I'_0 \quad (53) \\ \frac{\partial A^1}{\partial \sigma} &= I'_1 \sqrt{c_1^2 - c_3^2} \frac{(A^1 - \beta M)^{1/2}}{(A^1 + 3M)^{1/2}} \\ \frac{\partial^2 A^1}{\partial \sigma^2} &= \sqrt{c_1^2 - c_3^2} I''_1 \frac{(A^1 - \beta M)^{1/2}}{(A^1 + 3M)^{1/2}} \\ &+ (c_1^2 - c_3^2) I_1'^2 \frac{(3+\beta)M}{2(A^1 + 3M)^2}, \\ \frac{\partial A^5}{\partial \sigma} &= -I'_5 + c_3 I'_1 \frac{(A^1 + 3M)}{(A^1 - M)} \\ \frac{\partial^2 A^5}{\partial \sigma^2} &= -I''_5 + c_3 I''_1 \frac{(A^1 + 3M)}{(A^1 - M)} \\ &- c_3 \sqrt{c_1^2 - c_3^2} I_1'^2 \frac{4M}{(A^1 - M)^2} \frac{(A^1 - \beta M)^{1/2}}{(A^1 + 3M)^{1/2}}. \end{aligned}$$

The right hand sides of Eqs. (48) are therefore determined in terms of derivatives of  $I_1$  and  $I_5$ .

For convenience, we choose  $I'_0 = 0$ ,  $I_1 = \cos \sigma$  and  $I_5 = \sin \sigma$ . To simplify the equations of motion further, we fix  $\sigma = 0$ . Using these values of  $I_1$  and  $I_5$  and substituting the constraint Eq. (14) in Eqs. (48), we obtain

$$\begin{aligned} \ddot{B}^1 + \frac{c_1^2 4M B^1}{(r+3M)^3} &= -\sqrt{c_1^2 - c_3^2} \frac{(r - \beta M)^{1/2}}{(r+3M)^{1/2}} \quad (54) \\ &- \frac{2M(r-M)}{(r+3M)^3} \\ \ddot{B}^5 - \frac{4M}{(r-M)(r+3M)} \sqrt{\frac{(r-M)}{(r+3M)}} c_1^2 - c_3^2 \dot{B}^5 \\ &+ \frac{4M}{(r-M)^2} c_3 \dot{B}^1 + \frac{8M(r+M)}{(r-M)^3} c_3 \sqrt{\frac{(r-M)}{(r+3M)}} c_1^2 - c_3^2 \end{aligned}$$

$$= -c_3 \frac{(r + 3M)}{(r - M)}$$

The equations of motion are now ordinary differential equations and can be solved to find the first order corrections to the null string configuration. Although the equation for  $B^1$  is now decoupled, the numerical solution of these equations is highly nontrivial, even in the relatively simple case of a PGPS monopole. Moreover, in the above discussion, we have taken a particular value of  $\sigma$ . It would however be appropriate to make a more general ansatz for the functions  $I_0$ ,  $I_1$  and  $I_5$ , the above example being a first step.

The invariant string size is defined in Eq. (16) as,

$$dl^2 = X'^\mu X'^\nu G_{\mu\nu}(X) d\sigma^2. \quad (55)$$

Using Eq. (10), we have

$$\frac{dl^2}{d\sigma^2} = -\frac{1}{c^2} \dot{X}^\mu \dot{X}^\nu G_{\mu\nu}. \quad (56)$$

It is clear from the above that, using the above definition, the string length cannot be determined in the zeroth order, i.e., for the  $c \rightarrow 0$  limit. The first order correction to the invariant string length is given by

$$\Delta \left( \frac{dl^2}{d\sigma^2} \right) = -(\dot{A}^\mu \dot{B}^\nu + \dot{A}^\nu \dot{B}^\mu) G_{\mu\nu} \quad (57)$$

Using the constraint Eq. (14), the first order correction is

$$\Delta \left( \frac{dl^2}{d\sigma^2} \right) = A'^\mu A'^\nu G_{\mu\nu}(X) + G_{\mu\nu,\rho} \dot{A}^\mu \dot{A}^\nu B^\rho \quad (58)$$

In Ref. [7], the invariant string length is defined as

$$dl^2 = -\dot{X}^\mu \dot{X}^\nu d\sigma^2, \quad (59)$$

which ignores the factor  $1/c^2$  and is, therefore, not appropriate to describe null strings.

We calculate the string length for the simple case mentioned above. The first order correction to the invariant string length for the PGPS monopole is

$$\begin{aligned} \Delta \left( \frac{dl}{d\sigma} \right)^2 &= -c_1^2 I_1'^2 \frac{(r - \beta M)}{(r - M)} - c_3^2 I_1'^2 \frac{(3 + \beta)M}{(r - M)} \\ &+ c_3 I_5'^2 \frac{(r - \beta M)}{(r + 3M)} + 2I_0' I_5' \\ &+ \left( \frac{4M}{(r - M)(r + 3M)} c_1^2 - \frac{8M}{(r - M)^2} c_3^2 \right) B^1, \end{aligned} \quad (60)$$

which, for the choice of  $I_1$  and  $I_5$  given above and at the point  $\sigma = 0$ , reduces to

$$\begin{aligned} \left( \frac{dl}{d\sigma} \right)^2 &= c_3 I_5'^2 \frac{(r - \beta M)}{(r + 3M)} + \left( \frac{4M}{(r - M)(r + 3M)} c_1^2 \right. \\ &\left. - \frac{8M}{(r - M)^2} c_3^2 \right) B^1. \end{aligned} \quad (61)$$

Therefore, even with the simplistic approximations described above, there is a nontrivial contribution to the invariant string size which can be calculated once the first order equations are solved.

## VII. CONCLUSIONS

In this paper, the motion of a string near five-dimensional Kaluza-Klein black holes has been investigated. For simplicity, the electrically and magnetically charged cases have been considered separately. The equations of motion obtained from the string action are solved, numerically as well as analytically, to zeroth order in the null string expansion. The domain of interest is the region just outside the horizon and in the equatorial plane. In the magnetically charged black hole case the equations, although complicated, can be reduced to quadratures. The quadratures are solved analytically in terms of different combinations of elliptical integrals which depend on the relative values of the charges. There are some solutions where the elliptical integrals reduce to elementary functions. Detailed analytical as well as numerical investigation is carried out in a suitable large distance approximation, i.e. the scalar charge, in appropriate units, being much smaller than the distance from the black hole. However, for the electrically charged case, the equations cannot be reduced to quadratures and have to be solved numerically. The solutions express the string coordinates in terms of the proper time of the string. The extra coordinate behaves differently in the electric and magnetic cases. For a clearer picture, it is useful to define a quantity called the Kaluza-Klein radius, which is the radius of the circle around which the extra dimension winds. As the string approaches a magnetically charged black hole, the value of the Kaluza-Klein radius which it sees becomes smaller than its asymptotic value. The opposite effect is seen for an electrically charged black hole, where the extra dimension tends to expand.

One of the black hole solutions to five dimensional Kaluza-Klein theory is the well known Pollard-Gross-Perry-Sorkin (PGPS) monopole. It has been recently shown [20] that this monopole arises as a solution of a suitable dimensionally reduced superstring theory, thus providing an additional motivation for the present study. In this special case, the solutions to the string equation of motion are obtained analytically, even though the scalar charge is not small. The solution possesses an additional feature, namely, a decelerated fall of the string into the black hole. This can be understood as being due to the extremal nature of the black hole. The Kaluza-Klein theory possesses another extremal black hole solution not mentioned hitherto in the literature. The string equations of motion are solved in this background too and the solutions written in terms of elliptical integrals.

In the first order of the null string expansion, the effects of the background on the string become manifest in

the form of shape changes. The problem is technically very involved. An illustrative calculation for the PGPS monopole case is presented to show how one may proceed to set up the equations for numerical solution. Although the approximations made are simplistic, it is shown that the invariant string size receives a nonzero contribution in the first order. However, one should solve the second order partial differential equations, work on which is in progress.

Our approach is complementary to finding out stringy corrections to the five-dimensional vacuum Einstein equations and their effect on the black hole metrics [19]. We limit ourselves to studying only the classical picture. Although, in principle, quantum effects become dominant in the strong gravity regime, it seems plausible that the classical picture gives an intuitive idea of the compactification mechanism.

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### APPENDIX A: LEADING ORDER ANALYSIS OF EQUATIONS OF MOTION

This Appendix displays some details of the leading order analysis of string equations of motion in electrically charged Kaluza-Klein black hole backgrounds. The equations of motion for an electrically charged black hole background (Eqs. (25)), unlike those for a magnetic black hole, do not reduce to quadratures. One has to therefore resort to solving them numerically. To find out the initial conditions for numerical solutions, one has to analyse the equations to leading order in powers of  $1/r$ .

The equations of motion, in the limit where  $r \gg \Sigma_1$ , are given by Eqs. (28). Up to the leading order, the equations take the form

$$\begin{aligned} r^2 \frac{\partial^2 t}{\partial \tau^2} + 2(M + \Sigma_1) \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} + 4Q \frac{\partial r}{\partial \tau} \frac{\partial x_5}{\partial \tau} &= 0, \\ r^2 \frac{\partial^2 x_5}{\partial \tau^2} - 4Q \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} - 4\Sigma_1 \frac{\partial r}{\partial \tau} \frac{\partial x_5}{\partial \tau} &= 0, \\ r^2 \frac{\partial^2 r}{\partial \tau^2} + (\Sigma_1 + M) \left( \frac{\partial t}{\partial \tau} \right)^2 + 4Q \frac{\partial t}{\partial \tau} \frac{\partial x_5}{\partial \tau} \\ + (\Sigma_1 - M) \left( \frac{\partial r}{\partial \tau} \right)^2 + 2\Sigma_1 \left( \frac{\partial x_5}{\partial \tau} \right)^2 &= 0. \end{aligned} \quad (\text{A1})$$

with the constraint

$$\left( \frac{\partial r}{\partial \tau} \right)^2 = \left( \frac{\partial t}{\partial \tau} \right)^2 - \left( \frac{\partial x_5}{\partial \tau} \right)^2. \quad (\text{A2})$$

We make the reasonable assumption that the trajectories can be written in the parametric form  $t = t(r)$ ,  $\theta = \theta(r)$ ,  $\phi = \phi(r)$  and  $x_5 = x_5(r)$ . We can then change the derivatives w.r.t.  $\tau$  to  $r$  derivatives, i.e.,

$$\frac{\partial t}{\partial \tau} = \frac{dt}{dr} \frac{\partial r}{\partial \tau}, \quad \frac{\partial x_5}{\partial \tau} = \frac{dx_5}{dr} \frac{\partial r}{\partial \tau}. \quad (\text{A3})$$

We have (considering only the  $t$  and  $x_5$  equations for the time being)

$$\begin{aligned} r^2 \frac{d^2 t}{dr^2} - (\Sigma_1 + M) \left( \frac{dt}{dr} \right)^3 - 4Q \left( \frac{dt}{dr} \right)^2 \frac{dx_5}{dr} \\ - 2\Sigma_1 \left( \frac{dx_5}{dr} \right)^2 \frac{dt}{dr} + (\Sigma_1 + M) \frac{dt}{dr} + 4Q \frac{dx_5}{dr} &= 0, \\ r^2 \frac{d^2 x_5}{dr^2} - (\Sigma_1 + M) \left( \frac{dt}{dr} \right)^2 \frac{dx_5}{dr} - 4Q \frac{dt}{dr} \left( \frac{dx_5}{dr} \right)^2 \\ - (5\Sigma_1 - M) \frac{dx_5}{dr} - 2\Sigma_1 \left( \frac{dx_5}{dr} \right)^3 - 4Q \frac{dt}{dr} &= 0. \end{aligned} \quad (\text{A4})$$

with the constraint equation simplifying to

$$\left( \frac{dt}{dr} \right)^2 - \left( \frac{dx_5}{dr} \right)^2 = 1. \quad (\text{A5})$$

Defining

$$\begin{aligned} \psi_1 &\equiv \frac{dt}{dr} \\ \psi_2 &\equiv \frac{dx_5}{dr} \end{aligned} \quad (\text{A6})$$

the equations of motion are decoupled (Eqs. (30)) and are given by

$$\begin{aligned} r^2 \frac{d\psi_1}{dr} - (\Sigma_1 + M)\psi_1^3 + 3(\Sigma_1 + M)\psi_1 \\ \mp 4Q\sqrt{\psi_1^2 - 1}(\psi_1 - 1) &= 0, \\ r^2 \frac{d\psi_2}{dr} - 6\Sigma_1\psi_2^3 + (3\Sigma_1 + M)\psi_2 \\ \mp 4Q\sqrt{\psi_2^2 + 1}(\psi_2 + 1) &= 0, \end{aligned} \quad (\text{A7})$$

These equations are first order in derivatives of  $\psi_1$  and  $\psi_2$ . The equation for  $\psi_1$  can be solved to

$$\begin{aligned} \int \frac{d\psi_1}{(3\Sigma_1 + M)\psi_1^3 - 3(\Sigma_1 + M)\psi_1 - 4Q\sqrt{\psi_1^2 - 1}(\psi_1^2 - 1)} \\ = -\frac{1}{r} + I_c, \end{aligned} \quad (\text{A8})$$

where  $I_c$  is an integration constant.

The left hand side of the equation is an integral of the type

$$\int \frac{dx}{ax^3 + bx + c\sqrt{x^2 - 1}(x^2 - 1)}, \quad (\text{A9})$$

with  $a = 3\Sigma_1 + M$ ,  $b = -3(\Sigma_1 + M)$  and  $c = -4Q$ . The electric charge  $Q$  and the scalar charge  $\Sigma_1$  are related via the equation

$$Q^2 = \Sigma_1(\Sigma_1 + M). \quad (\text{A10})$$

With a change in variable  $x = \sec \theta$  followed by the substitution  $l = \sin \theta$ , and using Eq. (A10), the integral can be written as

$$\frac{1}{c} \int \frac{ldl}{l^3 - \frac{3}{2\epsilon}l^2 + \frac{1}{2\epsilon}}, \quad (\text{A11})$$

where  $\epsilon = \sqrt{\frac{2\Sigma_1}{M}}$  and  $\frac{b}{c} \approx \frac{3}{4\epsilon}$  and  $\frac{a+b}{c} \approx \frac{1}{2\epsilon}$ .

The term  $l^3$  is bounded between +1 and -1. We are interested in the limit of small scalar charge  $\Sigma_1$ , hence  $\frac{1}{\epsilon}$  is a large number. Therefore, the term  $l^3$  in the denominator can be neglected with respect to the terms with factors  $\propto \frac{1}{\epsilon}$ . Solving the integral and resubstituting for  $\psi_1$  we get the first equation in (31), viz.

$$\psi_1 = \frac{-1}{\sqrt{\mathcal{A}}\sqrt{2M/r + c_1}}, \quad (\text{A12})$$

where the negative sign is chosen for an in-falling string. A similar analysis for the equation for the coordinate  $x_5$  gives the expression for  $\psi_2$  which is given in Eqs. (31).

The equation of motion for  $r$  (up to the leading order) is given by

$$\begin{aligned} \frac{\partial^2 r}{\partial \tau^2} = & - \left\{ \frac{\Sigma_1 + M}{r^2} \left( \frac{dt}{dr} \right)^2 \right. \\ & + \frac{4Q}{r^2} \frac{dt}{dr} \frac{dx_5}{dr} + \frac{\Sigma_1 - M}{r^2} + \frac{2\Sigma_1}{r^2} \left( \frac{dx_5}{dr} \right)^2 \left. \right\} \left( \frac{dr}{d\tau} \right)^2. \end{aligned} \quad (\text{A13})$$

The leading order behaviour of the derivatives of  $t$  and  $x_5$  is substituted in the above equation and we have

$$\begin{aligned} \frac{\partial^2 r}{\partial \tau^2} = & - \left[ \frac{\Sigma_1 + M}{r^2} \frac{1}{\mathcal{A}(2M/r + c_1)} \right. \\ & + \frac{4Q}{r^2} \frac{1}{\sqrt{\mathcal{A}(2M/r + c_1)(2M/r + c_2)}} \\ & \left. - \frac{\Sigma_1 - M}{r^2} + \frac{2\Sigma_1}{r^2} + \frac{2\Sigma_1}{r^2} \frac{1}{(2M/r + c_2)} \right] \left( \frac{\partial r}{\partial \tau} \right)^2. \end{aligned} \quad (\text{A14})$$

The first integral of motion is then given by

$$\begin{aligned} \ln \left( \frac{\partial r}{\partial \tau} \right) = & - \left[ \frac{\Sigma_1 + M}{2\mathcal{A}M} \ln \left( \frac{2M}{r} + c_1 \right) \right. \\ & + \frac{\Sigma_1}{M} \ln \left( \frac{2M}{r} + c_2 \right) + \frac{M - \Sigma_1}{r} \\ & - \frac{4Q}{2\sqrt{\mathcal{A}M}} [\ln r - \ln(4M + c_1r + c_2r) \\ & \left. + 2\sqrt{(2M + c_1r)(2M + c_2r)}] \right] \end{aligned} \quad (\text{A15})$$

In the above equation,  $\Sigma_1/M$  and  $4Q/2\sqrt{\mathcal{A}M}$  are of  $O(\epsilon^2)$  and the terms can therefore be neglected. This implies,

$$\ln \left( \frac{\partial r}{\partial \tau} \right) \approx - \frac{1}{2\mathcal{A}} \ln \left( \frac{2M}{r} + c_1 \right) - \frac{M}{r}, \quad (\text{A16})$$

which can further be solved to

$$\frac{\partial r}{\partial \tau} = \frac{Ie^{-M/r}}{(2M/r + c_1)^{1/2\mathcal{A}}} \quad (\text{A17})$$

where  $I$  and  $c_1$  are integration constants. From the above values of  $dt/dr$ ,  $dx_5/dr$  and  $\partial r/\partial \tau$ , we can obtain Eqs. (33).

This equips us with the initial conditions for the numerical solutions of Eqs. (28). These equations can be solved to find the behaviour of  $t$  and  $x_5$  and the constraint equation can then be used to find out  $r$  as a function of  $\tau$ .

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- [1] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory* (Cambridge Univ. Press, 1987).
  - [2] J. Polchinski, *Superstring Theory* (Cambridge Univ. Press, 1998).
  - [3] H. J. De Vega and N. Sánchez, Lectures on String Theory in Curved Spacetimes, in: Proc. Third Paris Cosmology Colloquium (Paris, June 1995), ed. H. J. De Vega and N. Sánchez, (World Scientific, 1996).
  - [4] For a different approach, leading to possible observable consequences at particle accelerators, L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) and L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
  - [5] H. J. De Vega and A. Nicolaidis, Phys. Lett. **B295**, 241 (1992).
  - [6] H. J. De Vega, I. Giannakis and A. Nicolaidis, Mod. Phys. Lett. **A10**, 2479 (1995).
  - [7] C. O. Lousto and N. Sánchez, Phys. Rev. D **54**, 6399 (1996).
  - [8] J. M. Overduin and P. S. Wesson, Phys. Rep. **283** (1997) 303.
  - [9] H. K. Jassal, A. Mukherjee, and R. P. Saxena, Phys. Lett. B **462**, 55 (1999).
  - [10] H. K. Jassal and A. Mukherjee, Phys. Rev. D **62**, 067501 (2000).
  - [11] Although motivated by fundamental string theory, these methods are equally applicable to cosmic strings moving in black hole backgrounds as already mentioned in [5–7].
  - [12] A. Chodos and S. Detweiler, Gen. Rel. and Gravitation, **14**, 879 (1982).
  - [13] P. Dobaish and D. Maison, Gen. Rel. and Gravitation, **14**, 879 (1982).
  - [14] D. J. Gross and M. J. Perry, Nucl. Phys. **B226**, 29 (1983).
  - [15] D. Pollard, J. Phys. A **16**, 565 (1983).

- [16] R. D. Sorkin, Phys. Rev. Lett. **51**, 87 (1983).
- [17] G. W. Gibbons and D. L. Wiltshire, Ann. Phys. 167, **201** (1986); **176**, 393 (E) (1987).
- [18] M. Cvetič and D. Youm, Phys. Rev. Lett. **75**, 4165 (1995).
- [19] N. Itzhaki, Nucl. Phys. **B508**, 700 (1997).
- [20] S. Roy, Phys. Rev. D **60**, 082003 (1999).
- [21] H. J. De Vega and N. Sánchez, Phys. Lett. **B 197**, 320 (1987).
- [22] H. J. De Vega and N. Sánchez, Phys. Lett. **B 309**, 552, 557 (1988).
- [23] V. P. Frolov, V. D. Skarzinsky, A. I. Zelnikov and O. Heinrich, Phys. Lett. **B 224**, 225 (1989).
- [24] A. L. Larsen and N. Sánchez, Phys. Rev. D **51**, 6929 (1995).
- [25] H. J. De Vega and I. L. Egusquiza, Phys. Rev. D **54**, 7513 (1996).
- [26] V. Frolov, S. Hendy and J. P. De Villers, hep-th/9612199.
- [27] A. L. Larsen and N. Sánchez, Phys. Rev. D **54**, 2801 (1996).
- [28] S. Mahapatra, Phys. Rev. D **55**, 6403 (1997).
- [29] S. Kar and S. Mahapatra, Class. Quantum Grav. **15**, 421 (1998).
- [30] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, 1994).
- [31] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications Inc., 1964).
- [32] G. 't Hooft, Int. J. Mod. Phys. **A 11**, 4623 (1996).
- [33] G. W. Gibbons and P. K. Townsend, Phys. Lett. **B356**, 472 (1995).

